arXiv:0704.3627v4 [math.AG] 5 Jun 2010

THE SPECIAL MCKAY CORRESPONDENCE AS AN EQUIVALENCE OF DERIVED CATEGORIES

ALASTAIR CRAW

ABSTRACT. We give a new moduli construction of the minimal resolution of the singularity of type $\frac{1}{r}(1,a)$ by introducing the Special McKay quiver. To demonstrate that our construction trumps that of the G-Hilbert scheme, we show that the induced tautological line bundles freely generate the bounded derived category of coherent sheaves on X by establishing a suitable derived equivalence. This gives a moduli construction of the Special McKay correspondence for abelian subgroups of GL(2).

1. INTRODUCTION

For a finite subgroup $G \subset \mathrm{SL}(2,\Bbbk)$, the McKay correspondence establishes an equivalence between the geometry of the minimal resolution X of $\mathbb{A}^2_{\Bbbk}/G$ and the G-equivariant geometry of $\mathbb{A}^2_{\Bbbk}$. More precisely, following the description by Ito–Nakamura [8] of X as the G-Hilbert scheme, Kapranov–Vasserot [9] used the resulting universal family to establish an equivalence between the bounded derived category of coherent sheaves on X and the bounded derived category of finitely generated modules over the skew group algebra $\Bbbk[x,y] * G$. For a finite subgroup $G \subset \mathrm{GL}(2,\Bbbk)$ that is not special-linear, however, the G-Hilbert scheme has too many tautological bundles, so this moduli description of the McKay correspondence cannot hold without some redundancy. Nevertheless, Wunram [16] constructed an integral basis of the Grothendieck group of vector bundles on X that is indexed by the trivial representation and the so-called *special* representations of G. Ishii [7] subsequently employed the universal family for the G-Hilbert scheme to establish a fully faithful functor from the bounded derived category of compactly supported coherent sheaves on X to the bounded derived category of finitely generated nilpotent modules over $\Bbbk[x,y] * G$.

This article adopts a new approach for a finite abelian subgroup $G \subset \mathrm{GL}(2,\Bbbk)$ by introducing a new moduli construction of the minimal resolution X of $\mathbb{A}^2_{\Bbbk}/G$. To begin, we fix a collection of line bundles $L_0 = \mathscr{O}_X, L_1, \ldots, L_\ell$ on X that form an integral basis of the Grothendieck group of vector bundles. The algebra $\mathrm{End}\big(\bigoplus_{0\leq i\leq \ell} L_i\big)$ is isomorphic to the quotient $\Bbbk Q/R$ of the path algebra of a quiver Q modulo an ideal of relations R, where the pair (Q,R) is the bound quiver of sections of the collection $\underline{\mathscr{L}} = (L_0,\ldots,L_\ell)$ as defined by Craw–Smith [5]. Since (Q,R) can be characterised in terms of Wunram's special representations, we call (Q,R) the *bound Special McKay quiver*. The main result of this article is the following.

Theorem 1.1. *Let $G \subset \mathrm{GL}(2,\Bbbk)$ be a finite abelian subgroup with bound Special McKay quiver (Q,R). The minimal resolution X of $\mathbb{A}^2_{\Bbbk}/G$ is isomorphic to the fine moduli space $\mathcal{M}_\vartheta$ of ϑ-stable representations of (Q,R) for a given ϑ, with tautological bundle $\bigoplus_{0\leq i\leq\ell} L_i$.*

2000 *Mathematics Subject Classification.* Primary 14D22, 16G20, 18E30; Secondary 14M25, 14E15.

The proof of this result has two parts. The first, geometric part extends the construction of Craw–Smith [5] by defining the morphism $\varphi_{|\underline{\mathscr{L}}|}\colon X \longrightarrow |\underline{\mathscr{L}}|$ to the multigraded linear series associated to the sequence of line bundles $\underline{\mathscr{L}} = (L_0, \dots, L_\ell)$ on X. The toric variety $|\underline{\mathscr{L}}|$ is defined to be the fine moduli space of ϑ-stable representations of the quiver of sections Q of $\underline{\mathscr{L}}$ for a given dimension vector and stability condition ϑ, and it contains $\mathcal{M}_\vartheta(Q,R)$ as a closed subscheme. We prove that $\varphi_{|\underline{\mathscr{L}}|}$ is a closed immersion and, moreover, we identify explicitly the image as a toric subvariety $\mathbb{V}(I_Q)/\!\!/_\vartheta T_Q$ of $\mathcal{M}_\vartheta(Q,R)$.

The second, algebraic part establishes a link between the bound quiver of sections (Q,R) of $\underline{\mathscr{L}}$ and the bound McKay quiver of $G \subset \mathrm{GL}(2,\Bbbk)$. If the given two-dimensional representation of G decomposes into irreducible representations as $\mathbb{A}^2_\Bbbk = \rho_1 \oplus \rho_2$, then the McKay quiver Q' is the quiver whose vertices are indexed by the irreducible representations, and where each vertex ρ admits two incoming arrows: one arrow from vertex $\rho \otimes \rho_1$; and the second from vertex $\rho \otimes \rho_2$. The ideal of relations $R' \subset \Bbbk Q'$ ensures that $\Bbbk Q'/R'$ is isomorphic to the skew group algebra $\Bbbk[x,y] * G$, and the pair (Q', R') is the bound McKay quiver of $G \subset \mathrm{GL}(2,\Bbbk)$. We show that (Q', R') coincides with the bound quiver of sections of the tautological line bundles on the G-Hilbert scheme $G\text{-Hilb}(\mathbb{A}^2_\Bbbk)$ and, moreover, that the isomorphism between $G\text{-Hilb}(\mathbb{A}^2_\Bbbk)$ and the minimal resolution X identifies the tautological bundles on $G\text{-Hilb}(\mathbb{A}^2_\Bbbk)$ indexed by the trivial and special representations of G with the integral basis $L_0, L_1, \dots, L_\ell$ of the Grothendieck group of X. This implies in particular that $\mathrm{End}\big(\bigoplus_{0 \le i \le \ell} L_i\big)$ is isomorphic to a subalgebra of $\Bbbk Q'/R'$. This description, coupled with our understanding of the relations R' in the McKay quiver, provides just enough information about R to show that the moduli space $\mathcal{M}_\vartheta(Q,R)$ coincides with the image $\mathbb{V}(I_Q)/\!\!/_\vartheta T_Q$ of the morphism $\varphi_{|\underline{\mathscr{L}}|}$.

Theorem 1.1 provides a moduli description for the following derived category version of the *Special McKay correspondence* that is essentially due to Van den Bergh [13].

Theorem 1.2. *Let $\mathscr{V} := \bigoplus_{0 \le i \le \ell} L_i$ denote the tautological vector bundle on $\mathcal{M}_\vartheta(Q,R)$. Then*

$$\mathbf{R}\mathrm{Hom}(\mathscr{V}, -)\colon D^b\big(\mathrm{Coh}(X)\big) \to D^b\big(\mathrm{mod}(A)\big)$$

is a derived equivalence between the bounded derived category of coherent sheaves on X and the bounded derived category of finitely generated right modules over $A := \mathrm{End}(\bigoplus_{0 \le i \le \ell} L_i)$.

Corollary 1.3. *The algebra $\Bbbk Q/R$ is a minimal noncommutative resolution of $\mathbb{A}^2_\Bbbk/G$.*

Wemyss [14] gives a complementary, algebraic approach to the two main results in this paper that is closer in spirit to noncommutative geometry than the approach adopted here. The benefit of his approach is that the algebra $\Bbbk Q/R$ is constructed without prior knowledge of the minimal resolution; the drawback is that the ideal of relations R must be computed explicitly. We believe that both approaches have their merits.

Conventions Write $\Bbbk$ for an algebraically closed field of characteristic zero, $\Bbbk^\times$ for the one-dimensional algebraic torus over $\Bbbk$, and $\mathbb{N}$ for the semigroup of nonnegative integers. We do not assume that toric varieties are normal.

Acknowledgements. Special thanks to Alastair King and Michael Wemyss for pointing out errors in an earlier version. Thanks to Akira Ishii, Diane Maclagan and Greg Smith for useful discussions, and to Bill Crawley-Boevey for a helpful remark.

2. Background

2.1. Minimal resolution by toric geometry. For a finite abelian subgroup $G \subset \mathrm{GL}(2, \Bbbk)$ of order r, let $G^\vee = \mathrm{Hom}(G, \Bbbk^\times)$ denote the character group and $\mathrm{Irr}(G)$ the set of equivalence classes of irreducible representations. After killing quasireflections and changing coordinates if necessary, we may assume that G is the cyclic group of order r generated by the diagonal matrix $g = \mathrm{diag}(\omega, \omega^a)$, where ω is a primitive rth root of unity and $\gcd(a, r) = 1$. This action is said to be of type $\frac{1}{r}(1, a)$. The given representation of G decomposes as $\rho_1 \oplus \rho_2$, where $\rho_1(g) = \omega$ and $\rho_2(g) = \omega^a$. The induced G-action on $\Bbbk[x, y]$ satisfies $g \cdot x = \rho_1(g^{-1})x$ and $g \cdot y = \rho_a(g^{-1})y$, and we obtain a $G^\vee$-grading of $\Bbbk[x, y]$ via $\deg(x) = \rho_1$ and $\deg(y) = \rho_2$.

To construct $\mathbb{A}^2_\Bbbk/G$ and its minimal resolution by toric geometry, define $N := \mathbb{Z}^2 + \mathbb{Z} \cdot \frac{1}{r}(1, a)$ and $M := \mathrm{Hom}_\mathbb{Z}(N, \mathbb{Z})$. The nonnegative quadrant σ in $N \otimes_\mathbb{Z} \mathbb{Q}$ gives $\mathbb{A}^2_\Bbbk/G = \mathrm{Spec}\, \Bbbk[\sigma^\vee \cap M]$ with dense algebraic torus $T_M := \mathrm{Spec}\, \Bbbk[M]$. For the minimal resolution, expand

$$\frac{r}{a} = c_1 - \cfrac{1}{c_2 - \frac{1}{\cdots - \frac{1}{c_\ell}}}, \tag{2.1}$$

as a Jung-Hirzebruch continued fraction, giving integers $c_1, \dots, c_\ell \in \mathbb{Z}_{\geq 2}$. Define $(\beta_{\ell+1}, \alpha_{\ell+1}) = (0, r)$, $(\beta_\ell, \alpha_\ell) = (1, a)$, and $(\beta_{i-1}, \alpha_{i-1}) := c_i(\beta_i, \alpha_i) - (\beta_{i+1}, \alpha_{i+1})$ for $1 \leq i \leq \ell$, which implies $(\beta_0, \alpha_0) = (r, 0)$. Define rays $\tau_0, \tau_1, \dots, \tau_{\ell+1}$ in $N \otimes_\mathbb{Z} \mathbb{Q}$, where each τ_i has primitive generator $\frac{1}{r}(\beta_i, \alpha_i) \in N$. The fan Σ in $N \otimes_\mathbb{Z} \mathbb{Q}$ obtained by subdividing the cone σ by $\tau_1, \dots, \tau_\ell$ is the minimal resolution $f \colon X \to \mathbb{A}^2_\Bbbk/G$. For $1 \leq i \leq \ell$, let $D_i = X_{Star(\tau_i)} \cong \mathbb{P}^1$ denote the exceptional T_M-invariant divisor on X with toric coordinates $[x^{\alpha_i} : y^{\beta_i}]$. Write σ_i for the cone generated by τ_i and τ_{i+1}, so X is covered by charts $U_i = \mathrm{Spec}\, \Bbbk[M \cap \sigma_i^\vee] \cong \mathbb{A}^2_\Bbbk$ for $0 \leq i \leq \ell$, where

$$U_i = \mathrm{Spec}\, \Bbbk\left[\frac{x^{\alpha_{i+1}}}{y^{\beta_{i+1}}}, \frac{y^{\beta_i}}{x^{\alpha_i}}\right].$$

For a global description, write $\Sigma(1) = \{\tau_0, \tau_1, \dots, \tau_{\ell+1}\}$ for the set of rays in Σ, and write $\mathbb{N}^{\Sigma(1)}$ and $\mathbb{Z}^{\Sigma(1)}$ respectively for the semigroup and for the lattice generated by the T_M-invariant prime divisors. Since X is smooth there is a short exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^{\Sigma(1)} \xrightarrow{\deg} \mathrm{Pic}(X) \longrightarrow 0, \tag{2.2}$$

where $\deg(D) = \mathscr{O}_X(D)$. The total coordinate ring of X is the polynomial ring $\Bbbk[x_0, \dots, x_{\ell+1}]$ obtained as the semigroup algebra of $\mathbb{N}^{\Sigma(1)}$. The degree map endows $\Bbbk[x_0, \dots, x_{\ell+1}]$ with a $\mathrm{Pic}(X)$-grading, and the algebraic torus $\mathrm{Hom}(\mathrm{Pic}(X), \Bbbk^\times)$ acts on $\mathbb{A}^{\Sigma(1)}_\Bbbk = \mathrm{Spec}\, \Bbbk[x_0, \dots, x_{\ell+1}]$. For any line bundle $L \in \mathrm{Pic}(X)$, the L-graded piece $\Bbbk[x_0, \dots, x_{\ell+1}]_L$ is isomorphic to $H^0(X, L)$. In particular, if L is f-ample then X is isomorphic to the GIT quotient

$$\mathbb{A}^{\Sigma(1)}_\Bbbk /\!\!/_L \mathrm{Hom}(\mathrm{Pic}(X), \Bbbk^\times) := \mathrm{Proj}\left(\bigoplus_{j \geq 0} \Bbbk[x_0, \dots, x_{\ell+1}]_{L^j}\right). \tag{2.3}$$

This global description coincides with the quotient construction of X from Cox [2].

2.2. Representations of bound quivers. Let Q be a finite connected quiver with vertex set Q_0, arrow set Q_1, and maps $\mathrm{h}, \mathrm{t} \colon Q_1 \to Q_0$ indicating the vertices at the head and tail of each arrow. The characteristic functions $\chi_i \colon Q_0 \to \mathbb{Z}$ for $i \in Q_0$ and $\chi_a \colon Q_1 \to \mathbb{Z}$ for $a \in Q_1$ form the standard integral bases of the vertex space $\mathbb{Z}^{Q_0}$ and the arrows space $\mathbb{Z}^{Q_1}$ respectively.

The incidence map $\mathrm{inc}\colon \mathbb{Z}^{Q_1} \to \mathbb{Z}^{Q_0}$ defined by $\mathrm{inc}(\chi_a) = \chi_{\mathrm{h}(a)} - \chi_{\mathrm{t}(a)}$ has image equal to the sublattice $\mathrm{Wt}(Q) \subset \mathbb{Z}^{Q_0}$ of functions $\theta\colon Q_0 \to \mathbb{Z}$ satisfying $\sum_{i\in Q_0} \theta_i = 0$. A nontrivial path in Q is a sequence of arrows $p = a_1 \cdots a_k$ with $\mathrm{h}(a_j) = \mathrm{t}(a_{j+1})$ for $1 \leq j < k$. We set $\mathrm{t}(p) = \mathrm{t}(a_1), \mathrm{h}(p) = \mathrm{h}(a_k)$ and $\mathrm{supp}(p) = \{a_1, \ldots, a_k\}$. Each $i \in Q_0$ gives a trivial path e_i where $\mathrm{t}(e_i) = \mathrm{h}(e_i) = i$. The path algebra $\Bbbk Q$ is the $\Bbbk$-algebra whose underlying $\Bbbk$-vector space has a basis consisting of paths in Q, where the product of basis elements equals the basis element defined by concatenation of the paths if possible, or zero otherwise.

A representation of a quiver Q consists of a $\Bbbk$-vector space W_i for $i \in Q_0$ and a $\Bbbk$-linear map $w_a\colon W_{\mathrm{t}(a)} \to W_{\mathrm{h}(a)}$ for $a \in Q_1$. We often write W as shorthand for $\big((W_i)_{i\in Q_0}, (w_a)_{a\in Q_1}\big)$. We consider only representations with $\dim_\Bbbk(W_i) \leq 1$ for all $i \in Q_0$. A map of representations $W \to W'$ is a family $\psi_i\colon W_i \to W'_i$ of $\Bbbk$-linear maps for $i \in Q_0$ that are compatible with the structure maps, that is, $w'_a\psi_{\mathrm{t}(a)} = \psi_{\mathrm{h}(a)}w_a$ for all $a \in Q_1$. With composition defined componentwise, we obtain the abelian category of representations of Q. For any rational weight $\theta \in \mathrm{Wt}(Q) \otimes_\mathbb{Z} \mathbb{Q}$, a representation W satisfying $\dim_\Bbbk(W_i) = 1$ for all $i \in Q_0$ is θ-stable if every proper, nonzero subrepresentation $W' \subset W$ satisfies $\theta(W') := \sum_{i\in\mathrm{supp}(W')} \theta_i > 0$, where $\mathrm{supp}(W') := \{i \in Q_0 : W'_i \neq 0\}$. For θ-semistability, replace $>$ with $\geq$.

Let R be a two-sided ideal in $\Bbbk Q$ generated by differences of the form $p - q \in \Bbbk Q$ where p, q are paths with the same head and tail, each of which comprises at least two arrows. We do not assume that R is admissible, so $\Bbbk Q/R$ need not be of finite dimension over $\Bbbk$. The pair (Q, R) is an example of a *bound quiver*, also known as a quiver with relations. For any representation $W = \big((W_i)_{i\in Q_0}, (w_a)_{a\in Q_1}\big)$ of Q and for any nontrivial path $p = a_1 \cdots a_k$ in Q, the evaluation of W on p is the $\Bbbk$-linear map $w_p\colon W_{\mathrm{t}(p)} \to W_{\mathrm{h}(p)}$ defined by the composition $w_p = w_{a_1} \cdots w_{a_k}$. A representation of the bound quiver (Q, R) is a representation W of Q such that $w_p = w_q$ for all relations $p - q \in R$. The abelian category of finite-dimensional representations of (Q, R) is equivalent to the category of finitely-generated $\Bbbk Q/R$-modules.

2.3. **Moduli spaces of quiver representations.** Let Q be a finite connected quiver. The space of representations W of Q for which $\dim_\Bbbk(W_i) = 1$ for all $i \in Q_0$ is

$$\mathbb{A}^{Q_1}_\Bbbk := \mathrm{Spec}\big(\Bbbk[y_a : a \in Q_1]\big) \cong \bigoplus_{a\in Q_1} \mathrm{Hom}_\Bbbk(W_{\mathrm{t}(a)}, W_{\mathrm{h}(a)}).$$

The incidence map gives a $\mathrm{Wt}(Q)$-grading of the polynomial ring $\Bbbk[y_a : a \in Q_1]$ that induces an action of the algebraic torus $T_Q := \mathrm{Hom}(\mathrm{Wt}(Q), \Bbbk^\times)$ on $\mathbb{A}^{Q_1}_\Bbbk$, where $(t \cdot w)_a = t_{\mathrm{h}(a)} w_a t^{-1}_{\mathrm{t}(a)}$. For $\theta \in \mathrm{Wt}(Q)$, let $\Bbbk[y_a : a \in Q_1]_\theta$ denote the θ-graded piece. If every θ-semistable representation of Q is θ-stable then King [11, Proposition 5.3] proved that the GIT quotient

$$\mathcal{M}_\theta(Q) := \mathbb{A}^{Q_1}_\Bbbk /\!\!/_\theta T_Q = \mathrm{Proj}\left(\bigoplus\nolimits_{j\geq 0} \Bbbk[y_a : a \in Q_1]_{j\theta}\right) \tag{2.4}$$

is the fine moduli space of isomorphism classes of θ-stable representations of Q. The T_Q-equivariant vector bundle $\bigoplus_{i\in Q_0} \mathscr{O}_{\mathbb{A}^{Q_1}}$ on $\mathbb{A}^{Q_1}_\Bbbk$ descends to a tautological vector bundle $\bigoplus_{i\in Q_0} \mathscr{W}_i$ on $\mathcal{M}_\theta(Q)$. Our quivers will always have a distinguished vertex (denoted $0 \in Q_0$ or $\rho_0 \in Q'_0$), and we normalise the universal family on $\mathcal{M}_\theta(Q)$ by identifying T_Q with $\{(t_i)_{i\in Q_0} \in (\Bbbk^\times)^{Q_0} : t_0 = 1\}$; this gives $\mathscr{W}_0 \cong \mathscr{O}_{\mathcal{M}_\theta(Q)}$.

To construct moduli spaces of bound quiver representations, let (Q, R) be a bound quiver where R is generated by path differences $p - q \in \Bbbk Q$. The map sending a path p to the monomial

$y_p := \prod_{a\in\mathrm{supp}(p)} y_a \in \Bbbk[y_a : a \in Q_1]$ enables us to define the ideal

$$I_R := \big(y_p - y_q \in \Bbbk[y_a : a \in Q_1] : p - q \in R\big) \tag{2.5}$$

of relations in $\Bbbk[y_a : a \in Q_1]$. A point in $\mathbb{A}^{Q_1}_{\Bbbk}$ corresponds to a representation of (Q, R) if and only it lies in the subscheme $\mathbb{V}(I_R)$ cut out by I_R. The binomial ideal I_R is homogeneous with respect to the $\mathrm{Wt}(Q)$-grading, so $\mathbb{V}(I_R)$ is invariant under the action of T_Q. If every ϑ-semistable representation of Q is ϑ-stable then

$$\mathcal{M}_\vartheta(Q, R) := \mathbb{V}(I_R)/\!\!/_\vartheta T_Q = \mathrm{Proj}\Big(\bigoplus_{j\geq 0} \big(\Bbbk[y_a : a \in Q_1]/I_R\big)_{j\vartheta}\Big) \tag{2.6}$$

is the fine moduli space of isomorphism classes of ϑ-stable representations of (Q, R). The tautological bundle on $\mathcal{M}_\vartheta(Q, R)$ is obtained from that on $\mathcal{M}_\vartheta(Q)$ by restriction.

3. Minimal resolution via multigraded linear series

3.1. **Quivers of sections and multigraded linear series.** Let $\underline{\mathscr{E}} := \big(\mathscr{E}_0, \mathscr{E}_1, \dots, \mathscr{E}_m\big)$ be a sequence of distinct effective line bundles on the minimal resolution X of $\mathbb{A}^2_{\Bbbk}/G$, where $\mathscr{E}_0 := \mathscr{O}_X$ and $m > 0$. A T_M-invariant section $s \in H^0(X, \mathscr{E}_j \otimes \mathscr{E}_i^{-1}) = \mathrm{Hom}(\mathscr{E}_i, \mathscr{E}_j)$ is *irreducible* if it does not factor through some $\mathscr{E}_k$ with $k \neq i, j$. The *quiver of sections* of $\underline{\mathscr{E}}$ is the quiver Q in which the vertex set $Q_0 = \{0, \dots, m\}$ corresponds to the line bundles in $\underline{\mathscr{E}}$, and where the arrows from i to j correspond to the irreducible sections in $H^0(X, \mathscr{E}_j \otimes \mathscr{E}_i^{-1})$. For each arrow $a \in Q_1$, we write $\mathrm{div}(a) := \mathrm{div}(s) \in \mathbb{N}^{\Sigma(1)}$ for the divisor of zeroes of the defining section $s \in H^0(X, \mathscr{E}_j \otimes \mathscr{E}_i^{-1})$ and, more generally, for any path p in Q we call $\mathrm{div}(p) := \sum_{a\in\mathrm{supp}(p)} \mathrm{div}(a)$ the *labelling divisor*. The *ideal of relations* in the path algebra $\Bbbk Q$ is the two-sided ideal

$$R = \big\langle p - q \in \Bbbk Q : \mathrm{h}(p) = \mathrm{h}(q), \mathrm{t}(p) = \mathrm{t}(q), \mathrm{div}(p) = \mathrm{div}(q)\big\rangle \tag{3.1}$$

Following Craw–Smith [5] we call (Q, R) the *bound quiver of sections of* $\underline{\mathscr{E}}$.

Lemma 3.1. *Let (Q, R) be the bound quiver of sections of $\underline{\mathscr{E}}$. The quiver Q is connected and the quotient algebra $\Bbbk Q/R$ is isomorphic to* $\mathrm{End}\big(\bigoplus_{i\in Q_0} \mathscr{E}_i\big)$.

Proof. The quiver is connected since $H^0(X, \mathscr{E}_i) \neq 0$ for $0 \leq i \leq m$. The algebra isomorphism follows as in the proof of [5, Proposition 3.3]. □

Remark 3.2. Since X is projective over $\mathbb{A}^2_{\Bbbk}/G$ we have $H^0(\mathscr{O}_X) \cong \Bbbk[x, y]^G$, so directed cycles in Q based at $i \in Q_0$ correspond to G-invariant monomials in $\Bbbk[x, y]$. This makes the semiprojective situation different from the projective case; notably, the order $\mathscr{E}_1, \dots, \mathscr{E}_m$ is unimportant.

The *multigraded linear series* of $\underline{\mathscr{E}}$ is the variety $|\underline{\mathscr{E}}| := \mathbb{A}^{Q_1}_{\Bbbk}/\!\!/_\vartheta T_Q$, where Q is the quiver of sections of $\underline{\mathscr{E}}$ and where $\vartheta := (-m, 1, \dots, 1) \in \mathrm{Wt}(Q)$ is a specially chosen weight. The next result extends the construction of Craw–Smith [5, Proposition 3.9] to the semiprojective setting.

Proposition 3.3. *The multigraded linear series $|\underline{\mathscr{E}}|$ is isomorphic to the fine moduli space $\mathcal{M}_\vartheta(Q)$. This smooth toric variety is projective over the affine quotient $\mathbb{A}^{Q_1}_{\Bbbk}/T_Q$, and it is obtained as the geometric quotient of $\mathbb{A}^{Q_1}_{\Bbbk} \setminus \mathbb{V}(B_Q)$ by T_Q where*

$$B_Q = \Bigg(\prod_{a\in\mathrm{supp}(T)} y_a : T \textit{ is a spanning tree in } Q \textit{ with root at } 0 \in Q_0\Bigg).$$

Proof. To prove the first statement it is enough to prove that every ϑ-semistable representation is ϑ-stable. Let W be a ϑ-semistable representation of Q. If $W' \subset W$ is a proper nonzero subrepresentation then $\sum_{i\in\mathrm{supp}(W')}\vartheta_i \geq 0$. Since $\vartheta_i = 1$ for $i \neq 0$ and $\vartheta_0 = -m$ we have $W'_0 = 0$ and $\vartheta(W') > 0$, so W is ϑ-stable as required. A representation $W = \big((W_i)_{i\in Q_0}, (w_a)_{a\in Q_1}\big)$ is ϑ-stable if and only if every subrepresentation $W' \subset W$ with $W'_0 \neq 0$ has $W'_i \neq 0$ for all $i \neq 0$, which holds if and only if there is a spanning tree T with root at 0 such that $\prod_{a\in\mathrm{supp}(T)} w_a \neq 0$. The ideal B_Q therefore cuts out the ϑ-unstable locus. That the toric variety $\mathbb{A}^{Q_1}_{\Bbbk} /\!\!/_{\vartheta} T_Q$ is smooth follows from the fact that the incidence map of Q is totally unimodular as in [5, Proposition 3.8], and variation of GIT quotient $\vartheta \mapsto 0$ gives a projective morphism from $\mathbb{A}^{Q_1}_{\Bbbk} /\!\!/_{\vartheta} T_Q$ to $\mathbb{A}^{Q_1}_{\Bbbk}/T_Q$. $\square$

Since $\mathcal{M}_\vartheta(Q)$ is a fine moduli space, it follows that the multigraded linear series $|\underline{\mathscr{E}}|$ carries tautological line bundles $\mathscr{W}_0, \ldots, \mathscr{W}_m$ where $\mathscr{W}_0 \cong \mathscr{O}_{|\underline{\mathscr{E}}|}$.

3.2. **A morphism to the multilinear series.** We choose once and for all a preferred sequence of line bundles on X, namely, the sequence

$$\underline{\mathscr{L}} = \big(\mathscr{O}_X, L_1, \ldots, L_\ell\big), \tag{3.2}$$

such that $\deg(L_i|_{D_j}) = \delta_{ij}$ for $1 \leq i, j \leq \ell$. These line bundles are nef and hence globally generated; in explicit toric coordinates, for $0 \leq j \leq \ell$, the free $\mathscr{O}_{U_j}$-module $L_i|_{U_j}$ of rank one is generated by x^{α_i} if $i \leq j$, and y^{β_i} if $i > j$.

The multigraded linear series of $\underline{\mathscr{L}}$ is the fine moduli space $|\underline{\mathscr{L}}| := \mathcal{M}_\vartheta(Q)$, where Q is the quiver of sections of $\underline{\mathscr{L}}$ and $\vartheta = (-\ell, 1, \ldots, 1) \in \mathrm{Wt}(Q)$. The map $\Phi_Q \colon \Bbbk[y_a : a \in Q_1] \to \Bbbk[\mathbb{N}^{\Sigma(1)}]$ defined by $\Phi_Q(y_a) = x^{\mathrm{div}(a)}$ induces a morphism $(\Phi_Q)^* \colon \mathbb{A}^{\Sigma(1)}_{\Bbbk} \to \mathbb{A}^{Q_1}_{\Bbbk}$ that is equivariant with respect to the actions of $\mathrm{Hom}(\mathrm{Pic}(X), \Bbbk^\times)$ and $T_Q = \mathrm{Hom}(\mathrm{Wt}(Q), \Bbbk^\times)$. Since each L_i is globally generated, [5, Section 4] shows that $(\Phi_Q)^*$ descends to give a morphism $\varphi_{|\underline{\mathscr{L}}|} \colon X \longrightarrow |\underline{\mathscr{L}}|$. To describe the image, write $\pi := (\mathrm{inc}, \mathrm{div}) \colon \mathbb{Z}^{Q_1} \to \mathrm{Wt}(Q) \oplus \mathbb{N}^{\Sigma(1)}$ for the $\mathbb{Z}$-linear map sending χ_a to $\big(\chi_{\mathrm{h}(a)} - \chi_{\mathrm{t}(a)}, \mathrm{div}(a)\big)$ for $a \in Q_1$. Write $\mathbb{N}(Q)$ for the image under π of the subsemigroup $\mathbb{N}^{Q_1}$ generated by χ_a for $a \in Q_1$, and write $\Bbbk[\mathbb{N}(Q)]$ for its semigroup algebra. The projections $\pi_1 \colon \mathbb{N}(Q) \to \mathrm{Wt}(Q)$ and $\pi_2 \colon \mathbb{N}(Q) \to \mathbb{N}^{\Sigma(1)}$ fit in to the commutative diagram

$$\begin{tikzcd} \mathbb{N}(Q) \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & \mathrm{Wt}(Q) \arrow[d, "\mathrm{pic}"] \\ \mathbb{N}^{\Sigma(1)} \arrow[r, "\deg"] & \mathrm{Pic}(X) \end{tikzcd} \tag{3.3}$$

where $\mathrm{pic}(\sum_{i\in Q_0}\theta_i\chi_i) := \bigotimes_{i\in Q_0} L_i^{\theta_i}$. Since $\Bbbk[y_a : a \in Q_1]$ is the semigroup algebra of $\mathbb{N}^{Q_1}$, the map π induces a surjective maps of $\Bbbk$-algebras $\pi_* \colon \Bbbk[y_a : a \in Q_1] \to \Bbbk[\mathbb{N}(Q)]$ with kernel

$$I_Q := \big(y^u - y^v \in \Bbbk[y_a : a \in Q_1] ; u - v \in \mathrm{Ker}(\pi)\big). \tag{3.4}$$

The incidence map factors through $\mathbb{N}(Q)$ to give the map π_1, so the action of T_Q on $\mathbb{A}^{Q_1}_{\Bbbk}$ restricts to an action on the toric variety $\mathbb{V}(I_Q) = \mathrm{Spec}\,\Bbbk[\mathbb{N}(Q)]$ cut out by the toric ideal I_Q.

Theorem 3.4. *Let Q denote the quiver of sections of the sequence of bundles $\underline{\mathscr{L}}$ from* (3.2)*. The morphism $\varphi_{|\underline{\mathscr{L}}|} \colon X \to |\underline{\mathscr{L}}|$ is a closed immersion with image $\mathbb{V}(I_Q) /\!\!/_{\vartheta} T_Q$, and the tautological line bundles $\mathscr{W}_0, \ldots, \mathscr{W}_\ell$ on $|\underline{\mathscr{L}}|$ satisfy $\varphi^*_{|\underline{\mathscr{L}}|}(\mathscr{W}_i) = L_i$ for $0 \leq i \leq \ell$.*

Proof. The toric ideal I_Q is the $\mathrm{Wt}(Q)$-homogeneous part of $\mathrm{Ker}(\Phi_Q)$. Thus, just as $\mathrm{Ker}(\Phi_Q)$ cuts out the image of $(\Phi_Q)^*$, so I_Q cuts out the image of $\varphi_{|\mathcal{L}|}$ after passing to the quotient by the action of T_Q. Following [5, Theorem 1.1], the image of $\varphi_{|\mathcal{L}|}$ is the GIT quotient $\mathbb{V}(I_Q)/\!\!/_{\vartheta} T_Q$.

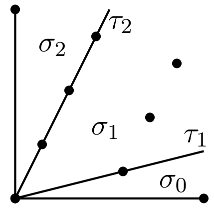
To see that $\varphi_{|\underline{\mathscr{L}}|}$ is a closed immersion, set $\mathcal{V} := \mathbb{N}(Q) \cap \pi_1^{-1}(\vartheta)$ and $\mathcal{W} := \mathbb{N}^{\Sigma(1)} \cap \deg^{-1}(L)$ for the very ample line bundle $L = \bigotimes_{i=1}^{\ell} L_i$. Since $L = \mathrm{pic}(\vartheta)$, we have $\pi_2(\mathcal{V}) \subseteq \mathcal{W}$ and hence $\Phi_Q(\Bbbk[y_a : a \in Q_1]_{\vartheta}) \subseteq \Bbbk[x_0, \dots, x_{\ell+1}]_L \cong H^0(X, L)$. According to the proof of [5, Theorem 4.9], the morphism $\varphi_{|\underline{\mathscr{L}}|}$ is a closed immersion if and only if the linear series $\Phi_Q(\Bbbk[y_a : a \in Q_1]_{\vartheta})$ defines a closed immersion. As a result, we must show that the $\Bbbk$-vector space basis $\pi_2(\mathcal{V})$ of $\Phi_Q(\Bbbk[y_a : a \in Q_1]_{\vartheta})$ contains the vertices $\{u(\sigma_j) : 0 \leq j \leq \ell\}$ of the polyhedron P_L obtained as the convex hull of the set $\mathcal{W}$ and, in addition, that the semigroup $M \cap \sigma_j^{\vee}$ is generated by $\{u - u(\sigma_j) : u \in \pi_2(\mathcal{V})\}$ for each $0 \leq j \leq \ell$. First, identify $\mathbb{N}(Q) \cap \pi_1^{-1}(\chi_i - \chi_0)$ with the monomial basis of $H^0(X, L_i)$ for each $0 \leq i \leq \ell$, and write $v_j(i) \in \mathbb{N}(Q) \cap \pi_1^{-1}(\chi_i - \chi_0)$ for the element $x^{\alpha_i} \in H^0(X, L_i)$ if $0 \leq i \leq j$, and $y^{\beta_i} \in H^0(X, L_i)$ if $j < i \leq \ell$. Since $L|_{U_j}$ is generated as an $\mathscr{O}_{U_j}$-module by $s_j := \prod_{0 \leq i \leq j} x^{\alpha_i} \cdot \prod_{j < i \leq \ell} y^{\beta_i}$, the element $v_j := \sum_{i=0}^{\ell} v_j(i) \in \mathcal{V}$ satisfies $\pi_2(v_j) = \mathrm{div}(s_j) = u(\sigma_j)$, so the vertices of P_L lie in $\pi_2(\mathcal{V})$. For the second part, write $v_{\alpha_i}, v_{\beta_i} \in \mathbb{N}(Q) \cap \pi_1^{-1}(\chi_i - \chi_0)$ for the elements $x^{\alpha_i}, y^{\beta_i} \in H^0(X, L_i)$ respectively, where $i = j, j+1$. Define elements of $\mathcal{V}$ by setting $v_j^+ := v_j + (v_{\beta_j} - v_{\alpha_j})$ and $v_j^- := v_j + (v_{\alpha_{j+1}} - v_{\beta_{j+1}})$. We obtain $\pi_2(v_j^+) = \mathrm{div}(s_j \cdot (y^{\beta_j}/x^{\alpha_j}))$ and $\pi_2(v_j^-) = \mathrm{div}(s_j \cdot (x^{\alpha_{j+1}}/y^{\beta_{j+1}}))$, so for each j the differences $v_j^+ - v_j, v_j^- - v_j \in \mathcal{V} - v_j$ map under π_2 to the generators $\mathrm{div}(y^{\beta_j}/x^{\alpha_j}), \mathrm{div}(x^{\alpha_{j+1}}/y^{\beta_{j+1}})$ of $M \cap \sigma_j^{\vee}$. Thus, the morphism $\varphi_{|\underline{\mathscr{L}}|}$ is indeed a closed immersion. For the statement about the tautological bundles, note that the proof of [5, Theorem 4.15] applies verbatim in this case. $\square$

Corollary 3.5. *The minimal resolution $f\colon X \to \mathbb{A}^2_{\Bbbk}/G$ coincides with the projective morphism $\mathbb{V}(I_Q)/\!\!/_{\vartheta} T_Q \to \mathbb{V}(I_Q)/T_Q$ obtained by variation of GIT quotient.*

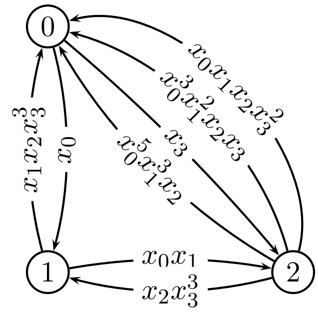
Proof. This is similar to the proof of [3, Proposition 4.1, Theorem 4.3(i)]. $\square$

Example 3.6. For the action of type $\frac{1}{7}(1,2)$, the rational curves D_1 and D_2 in X have toric coordinates $[x : y^4]$ and $[x^2 : y]$ respectively, as is evident from the fan of X shown in Figure 1(a). The sequence is $\underline{\mathscr{L}} = (\mathscr{O}_X, L_1, L_2)$, where: $L_1|_{U_j}$ is generated by x for $j = 1, 2$ and y^4 for $j = 0$; while $L_2|_{U_j}$ is generated by x^2 for $j = 2$ and y for $j = 0, 1$. Since L_1 has degree 1 on D_1 and degree 0 on D_2, we have $L_1 = \mathscr{O}_X(D_0)$ and, similarly, $L_2 = \mathscr{O}_X(D_3)$. The quiver of sections is shown in Figure 1(b), where each labelling divisor is recorded as a monomial in the Cox ring $\Bbbk[x_0, x_1, x_2, x_3]$. One approach to drawing the quiver of sections of $\underline{\mathscr{L}}$ is to first draw the quiver of sections of $\mathscr{O}_X$; that is, one vertex and five loops, each labelled with a $\Bbbk$-algebra generator of $H^0(\mathscr{O}_X)$. Adding a vertex for L_1 and then L_2 causes the loops to decompose into paths according to the decomposition of the labelling divisor. In this case, the semigroup $\mathbb{N}(Q)$ is generated by the columns of the matrix

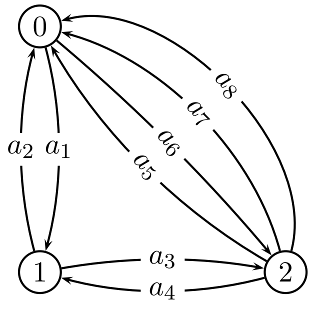
$$\begin{bmatrix} -1 & 1 & 0 & 0 & 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & -1 & -1 \\ 1 & 0 & 1 & 0 & 5 & 0 & 3 & 1 \\ 0 & 1 & 1 & 0 & 3 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 3 & 0 & 3 & 0 & 1 & 1 & 2 \end{bmatrix},$$



(a) Toric fan



(b) Quiver of sections



(c) Listing the arrows

FIGURE 1. A quiver of sections for the action of type $\frac{1}{7}(1,2)$

where the i-th column corresponds to a_i for $1 \leq i \leq 8$. One computes using Macaulay2 [6] that

$$I_Q = \begin{pmatrix} y_7^2 - y_5y_8,\ y_3y_4 - y_6y_8,\ y_1y_2 - y_6y_8,\ y_3y_7y_8 - y_2y_5,\ y_1y_7y_8 - y_4y_5, \\ y_1y_3y_7 - y_5y_6,\ y_3y_8^2 - y_2y_7,\ y_1y_8^2 - y_4y_7,\ y_1y_3y_8 - y_6y_7 \end{pmatrix}$$

is the toric ideal that cuts out the image of X under the morphism $\varphi_{|\mathscr{L}|}\colon X \to |\mathscr{L}|$.

Remark 3.7. It is easy to see that the ideal I_R arising from (Q, R) as in (2.5) is contained in I_Q, so the image $\mathbb{V}(I_Q)/\!\!/_{\vartheta} T_Q$ of the closed immersion $\varphi_{|\mathscr{L}|}\colon X \to |\mathscr{L}|$ is a subvariety of the fine moduli space $\mathcal{M}_\vartheta(Q, R)$ of ϑ-stable representations of the bound quiver (Q, R).

4. THE BOUND SPECIAL MCKAY QUIVER

4.1. **The bound McKay quiver.** The *McKay quiver* of the G-action of type $\frac{1}{r}(1, a)$ is the quiver Q' with vertex set $Q'_0 = \operatorname{Irr}(G)$ and arrow set $Q'_1 = \{a_1^\rho, a_2^\rho : \rho \in \operatorname{Irr}(G)\}$, where arrow a_i^ρ goes from vertex $\rho\rho_i := \rho \otimes \rho_i$ to vertex ρ for all $\rho \in \operatorname{Irr}(G)$ and $i = 1, 2$. The *label* of each arrow a_1^ρ is the monomial $\operatorname{mon}(a_1^\rho) = x$, and similarly, the label on a_2^ρ is $\operatorname{mon}(a_2^\rho) = y$. More generally, the label of a path p' in Q' is the product $\operatorname{mon}(p') = \prod_{a' \in \operatorname{supp}(p')} \operatorname{mon}(a')$. Let $\Bbbk Q'$ denote the path algebra of the quiver Q', and consider the two-sided ideal in $\Bbbk Q'$ given by

$$R' := \big\langle a_2^{\rho\rho_1} a_1^\rho - a_1^{\rho\rho_2} a_2^\rho : \rho \in \operatorname{Irr}(G) \big\rangle \tag{4.1}$$

Equivalently, R' is generated by path differences $p' - q' \in \Bbbk Q$ for which p', q' have the same head, tail and labelling monomial. The pair (Q', R') is the McKay quiver with relations, or equivalently, the *bound McKay quiver*, of the subgroup $G \subset \operatorname{GL}(2, \Bbbk)$.

The McKay correspondence provides a strong link between the bound McKay quiver and the minimal resolution X of $\mathbb{A}^2_\Bbbk/G$. To state the result, let $\{\chi_\rho : \rho \in \operatorname{Irr}(G)\}$ denote the standard basis of the vertex space $\mathbb{Z}^{\operatorname{Irr}(G)}$ of Q', where $\rho_0 \in \operatorname{Irr}(G)$ denotes the trivial representation.

Lemma 4.1. *Let (Q', R') denote the bound McKay quiver and set $\vartheta' := \sum_{\rho \in \operatorname{Irr}(G)} (\chi_\rho - \chi_{\rho_0})$. The fine moduli space $\mathcal{M}_{\vartheta'}(Q', R')$ of ϑ'-stable representations is isomorphic to X. Moreover, if $\bigoplus_{\rho \in \operatorname{Irr}(G)} \mathscr{W}_\rho$ denotes the tautological bundle on $\mathcal{M}_{\vartheta'}(Q', R')$, then the following are isomorphic:*

(i) *the skew group algebra* $\Bbbk[x, y] * G$*;*
(ii) *the quotient algebra* $\Bbbk Q'/R'$*; and*
(iii) *the endomorphism algebra* $\operatorname{End}\big(\bigoplus_{\rho \in \operatorname{Irr}(G)} \mathscr{W}_\rho\big)$.

Proof. The fine moduli space $\mathcal{M}_{\vartheta'}(Q',R')$ is isomorphic to the G-Hilbert scheme $G\text{-Hilb}(\mathbb{A}^2_{\Bbbk})$, so the first statement follows from Kidoh [10]. For the second, the isomorphism between (i) and (ii) is well known (see [4]). For the isomorphism between (ii) and (iii), let $\Gamma \subset \mathrm{SL}(3,\Bbbk)$ be the subgroup of type $\frac{1}{r}(1,a,r-a-1)$ whose McKay quiver Q_Γ can be constructed from that of $G \subset \mathrm{GL}(2,\Bbbk)$ by adding an arrow labelled z from ρ to $\rho\rho_1\rho_2$ for all $\rho \in \mathrm{Irr}(\Gamma) = \mathrm{Irr}(G)$, and the corresponding ideal of relations $R_\Gamma \subset \Bbbk Q_\Gamma$ includes additional generators to ensure that arrows labelled z commute with those labelled x and those labelled y. The McKay correspondence for Γ by Bridgeland–King–Reid [1] gives $\Bbbk Q_\Gamma/R_\Gamma \cong \mathrm{End}_{\mathscr{O}_Y}\left(\bigoplus_{\rho\in\mathrm{Irr}(\Gamma)}\mathscr{R}_\rho\right)$, where $\bigoplus_{\rho\in\mathrm{Irr}(\Gamma)}\mathscr{R}_\rho$ is the tautological bundle on $Y = \Gamma\text{-Hilb}(\mathbb{A}^3_{\Bbbk})$. Since the G-action on $\mathbb{A}^2_{\Bbbk}$ is obtained from the Γ-action on $\mathbb{A}^3_{\Bbbk}$ by setting $z=0$, functoriality of the G-Hilbert scheme implies that $X = G\text{-Hilb}(\mathbb{A}^2_{\Bbbk})$ is a divisor of $Y = \Gamma\text{-Hilb}(\mathbb{A}^3_{\Bbbk})$, and the tautological bundle $\bigoplus_{\rho\in\mathrm{Irr}(G)}\mathscr{W}_\rho$ on X is obtained from that on Y by setting $z=0$ throughout. The result follows since $\Bbbk Q/R$ is obtained from $\Bbbk Q_\Gamma/R_\Gamma$ by setting to zero all arrows labelled z. $\square$

The fine moduli space $\mathcal{M}_{\vartheta'}(Q',R')$ represents the same functor as the G-Hilbert scheme, so the tautological bundles $\{\mathscr{W}_\rho : \rho \in \mathrm{Irr}(G)\}$ on $\mathcal{M}_{\vartheta'}(Q',R')$ coincide with those on $G\text{-Hilb}(\mathbb{A}^2_{\Bbbk})$. To characterise these bundles, Kidoh [10, Theorem 5.1] observed that for $0 \le j \le \ell$ and for the toric chart $U_j = \operatorname{Spec}\Bbbk[x^{\alpha_{j+1}}/y^{\beta_{j+1}}, y^{\beta_j}/x^{\alpha_j}]$ on X, the unique standard monomial of the ideal $I_j = (x^{\alpha_{j+1}}, y^{\beta_j}, x^{\alpha_{j+1}-\alpha_j}y^{\beta_j-\beta_{j+1}})$ in degree $\rho \in \mathrm{Irr}(G)$ generates the rank one $\mathscr{O}_{U_j}$-module $\mathscr{W}_{\rho^*}|_{U_j}$, where ρ^* is the contragradient representation (recall from Section 2.1 that $\rho := \deg(x^iy^j)$ satisfies $\rho(g) = \omega^{i+aj}$). In particular, each $\mathscr{W}_\rho$ is nef.

Remark 4.2. We emphasise that the monomials defining sections of the tautological bundle $\mathscr{W}_{\rho^*}$ on $\mathcal{M}_{\vartheta'}(Q',R')$ indexed by vertex $\rho^* \in \mathrm{Irr}(G)$ have degree $\rho \in \mathrm{Irr}(G)$.

Proposition 4.3. *The bound McKay quiver (Q',R') is the bound quiver of sections for the sequence of tautological line bundles $\underline{\mathscr{L}}' := (\mathscr{W}_\rho : \rho \in \mathrm{Irr}(G))$ on $\mathcal{M}_{\vartheta'}(Q',R') \cong X$.*

Proof. Each tautological line bundle $\mathscr{W}_\rho$ is nef and $\mathscr{W}_{\rho_0} \cong \mathscr{O}_X$, so the bound quiver of sections $(Q_{\underline{\mathscr{L}}'}, R_{\underline{\mathscr{L}}'})$ for $\underline{\mathscr{L}}'$ is well defined. The vertex sets of Q' and $Q_{\underline{\mathscr{L}}'}$ coincide. For any closed point $[W] \in \mathcal{M}_{\vartheta'}(Q',R')$, the fibre $\mathscr{W}_\rho|_{[W]}$ is the $\Bbbk$-vector space W_ρ encoded in the representation of Q' parametrised by $[W]$. As the point $[W]$ varies in $\mathcal{M}_{\vartheta'}(Q',R')$, the maps $w_i^\rho \colon W_{\rho\rho_i} \to W_\rho$ arising from arrows $a_i^\rho \in Q'_1$ determine sections $s_i^\rho \in \mathrm{Hom}(\mathscr{W}_{\rho\rho_i}, \mathscr{W}_\rho) = H^0(\mathscr{W}_\rho \otimes \mathscr{W}_{\rho\rho_i}^{-1})$. These sections are irreducible because R' lies in the ideal of $\Bbbk Q'$ generated by paths of length two. Each arrow in Q' therefore determines an arrow in $Q_{\mathcal{L}'}$ with the same head and tail, so Q' is a subquiver of $Q_{\mathcal{L}'}$. For $a_2^{\rho\rho_1}a_1^\rho - a_1^{\rho\rho_2}a_2^\rho \in R'$, the relation $w_2^{\rho\rho_1}w_1^\rho = w_1^{\rho\rho_2}w_2^\rho \colon W_{\rho\rho_1\rho_2} \to W_\rho$ determines a relation $s_2^{\rho\rho_1}s_1^\rho = s_1^{\rho\rho_2}s_2^\rho \colon \mathscr{W}_{\rho\rho_1\rho_2} \to \mathscr{W}_\rho$ between sections. Thus, R' is a subset of $R_{\mathcal{L}'}$ under the inclusion of $\Bbbk Q'$ as a subalgebra of $\Bbbk Q_{\mathcal{L}'}$. Thus far we have $Q' \subseteq Q_{\mathcal{L}'}$ and $R' \subseteq R_{\mathcal{L}'}$. Lemma 3.1 and Lemma 4.1 together give an isomorphism $\Bbbk Q'/R' \cong \Bbbk Q_{\mathcal{L}'}/R_{\mathcal{L}'}$. If there exists an arrow $a \in Q_{\mathcal{L}'} \setminus Q'$ then this algebra isomorphism forces a relation in $R_{\mathcal{L}'}$ that contains a term of the form $\lambda a + f$ for some $\lambda \in \Bbbk$ and $f \in \Bbbk Q_{\mathcal{L}'}$. However, $R_{\mathcal{L}'}$ is an ideal of relations in a quiver of sections and hence every generating relation is a combination of paths of length at least two, so $Q' = Q_{\mathcal{L}'}$ after all. In particular, $\Bbbk Q' = \Bbbk Q_{\mathcal{L}'}$. The isomorphism $\Bbbk Q'/R' \cong \Bbbk Q_{\mathcal{L}'}/R_{\mathcal{L}'}$ now forces the inclusion $R' \subseteq R_{\mathcal{L}'}$ to be equality as required. $\square$

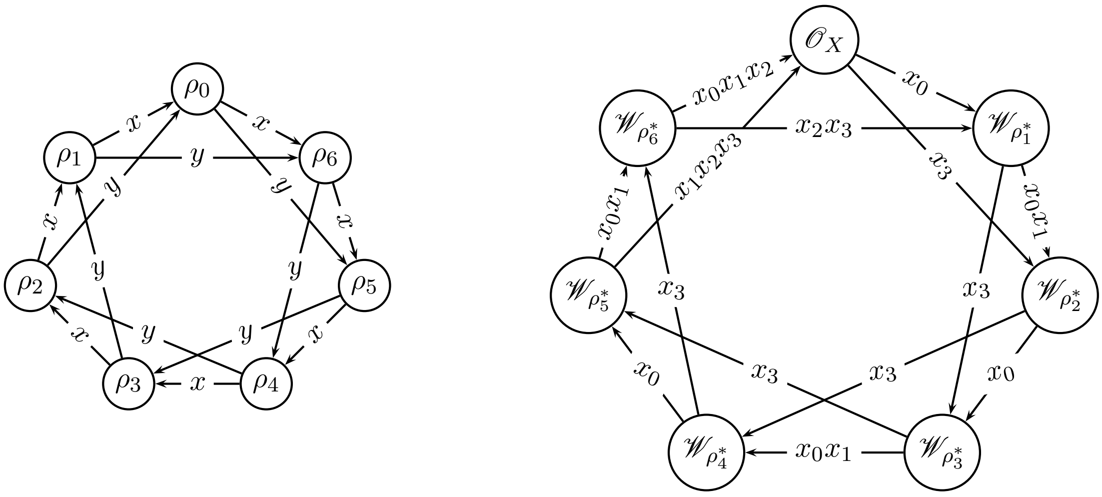
Corollary 4.4. *For paths p', q' in Q' with the same head and tail, the following are equivalent:*

(i) $\operatorname{div}(p') = \operatorname{div}(q') \in \Bbbk[x_0, \ldots, x_{\ell+1}]$.

(ii) $p' - q' \in R'$;

(iii) $\operatorname{mon}(p') = \operatorname{mon}(q') \in \Bbbk[x, y]$;

Example 4.5. Figure 2(a) shows the McKay quiver for the action of type $\frac{1}{7}(1, 2)$ with labels x and y, while Figure 2(b) gives the labels on Q' as a quiver of sections. Replace x_0 by x, x_3 by y and set $x_1 = x_2 = 1$ to recover the labels in Figure 2(a) from those in Figure 2(b).



(a) (b)

FIGURE 2. Labels on arrows in the McKay quiver of type $\frac{1}{7}(1, 2)$

4.2. **The bound Special McKay quiver.** For $\rho \in \operatorname{Irr}(G)$, let $(\Bbbk[x, y] \otimes_{\Bbbk} \rho^*)^G$ denote the $\Bbbk[x, y]^G$-submodule of $\Bbbk[x, y]$ generated by monomials of degree ρ. Following Wunram [15], we say that $\rho^* \in \operatorname{Irr}(G)$ is *special* if $(\Bbbk[x, y] \otimes_{\Bbbk} \rho^*)^G$ has two minimal $\Bbbk[x, y]^G$-module generators. We define the *bound Special McKay quiver* of $G \subset \operatorname{GL}(2, \Bbbk)$ to be the bound quiver of sections of the sequence of line bundles $\big(\mathscr{W}_{\rho^*} : \rho^* \in \operatorname{Irr}^{\mathrm{sp}}(G)\big)$ on $\mathcal{M}_{\vartheta'}(Q', R') \cong X$ where $\operatorname{Irr}^{\mathrm{sp}}(G)$ denotes the union of the trivial representation with the special representations.

Lemma 4.6. *The bound Special McKay quiver coincides with the bound quiver of sections (Q, R) of the sequence $\underline{\mathscr{L}}$ from* (3.2).

Proof. A representation ρ^* is special if and only if there is a prime exceptional divisor D_i in X with toric coordinates $[x^{\alpha_i} : y^{\beta_i}]$ such that $\rho = \deg(x^{\alpha_i}) = \deg(y^{\beta_i})$. For a special representation and for $0 \leq j \leq \ell$, the free $\mathscr{O}_{U_j}$-module $\mathscr{W}_{\rho^*}|_{U_j}$ of rank one is generated by x^{α_i} if $i \leq j$, and y^{β_i} if $i > j$. It follows that the tautological line bundles $\mathscr{W}_{\rho^*}$ indexed by special representations ρ^* coincide with the line bundles $L_1, \ldots, L_\ell$ satisfying $\deg(L_i|_{D_j}) = \delta_{ij}$ for $1 \leq i, j \leq \ell$. In particular, the sequence $\big(\mathscr{W}_{\rho^*} : \rho^* \in \operatorname{Irr}^{\mathrm{sp}}(G)\big)$ coincides with the sequence $\underline{\mathscr{L}}$ from (3.2). $\square$

The next result describes the link between the bound McKay and Special McKay quivers.

Proposition 4.7. *The following algebras are isomorphic:*

(i) *the quotient algebra* $\Bbbk Q/R$ *for the bound Special McKay quiver* (Q, R)*;*
(ii) *the endomorphism algebra* $\operatorname{End}\big(\bigoplus_{0\leq i\leq \ell} L_i\big)$*;*
(iii) *the endomorphism algebra* $\operatorname{End}\big(\bigoplus_{\rho^*\in \operatorname{Irr}^{\rm sp}(G)} \mathscr{W}_{\rho^*}\big)$*;*
(iv) *the algebra* $e(\Bbbk Q'/R')e$ *for the bound McKay quiver* (Q', R') *where* $e = \sum_{\rho^*\in \operatorname{Irr}^{\rm sp}(G)} e_{\rho^*}$*.*

Proof. Since Q is the bound quiver of sections of $\underline{\mathscr{L}}$, the isomorphism (i) $\cong$ (ii) follows from Lemma 3.1. Each L_i coincides with $\mathscr{W}_{\rho^*}$ for a unique $\rho^* \in \operatorname{Irr}^{\rm sp}(G)$, which gives the isomorphism between (ii) and (iii). Lemma 4.1(ii) implies that the subalgebra $e(\Bbbk Q'/R')e$ is isomorphic to $\operatorname{End}\big(\bigoplus_{\rho^*\in \operatorname{Irr}^{\rm sp}(G)} \mathscr{W}_{\rho^*}\big)$ which gives the isomorphism (iii) $\cong$ (iv). $\square$

The assignment sending a vertex $i \in Q_0$ to the representation $\rho^* \in \operatorname{Irr}(G)$ for which $L_i = \mathscr{W}_{\rho^*}$ establishes a bijection $\iota\colon Q_0 \to \operatorname{Irr}^{\rm sp}(G)$. Proposition 4.7 implies that for every arrow $a \in Q_1$ there is a path $p' := \iota(p)$ in Q' satisfying $\mathrm{h}(p') = \iota(\mathrm{h}(a))$, $\mathrm{t}(p') = \iota(\mathrm{t}(a))$ and $\operatorname{div}(p') = \operatorname{div}(a)$. The path $\iota(p)$ is not unique in general, though any two choices differ by an element of the ideal of relations R'. The situation is similar in the opposite direction: for any path p' in Q' with $\mathrm{t}(p'), \mathrm{h}(p') \in \operatorname{Irr}^{\rm sp}(G)$, there is a path $p := \lambda(p')$ in Q satisfying $\mathrm{h}(p') = \iota(\mathrm{h}(p))$, $\mathrm{t}(p') = \iota(\mathrm{t}(p))$ and $\operatorname{div}(p') = \operatorname{div}(p)$; again, $\lambda(p')$ is not unique in general, though any two choices differ by an element of the ideal R. In particular, $\iota(\lambda(p')) - p' \in R'$ and $\lambda(\iota(p)) - p \in R$.

Corollary 4.8. *For paths p, q in Q with the same head and tail, the following are equivalent:*
(i) $\operatorname{div}(p) = \operatorname{div}(q) \in \Bbbk[x_0, \dots, x_{\ell+1}]$.
(ii) $p - q \in R$*;*
(iii) $\operatorname{mon}(p) = \operatorname{mon}(q) \in \Bbbk[x, y]$*;*

Proof. This follows from Corollary 4.4 and Proposition 4.7. $\square$

We now turn our attention to understanding the arrow set of Q, and to begin we cite a useful combinatorial result. It is well known that the minimal $\Bbbk$-algebra generators of $\Bbbk[x, y]^G$ can be written in terms of the continued fraction expansion of $r/(r-a)$, but it is more convenient for us to work with the expansion of r/a from (2.1) which encodes the nonnegative integers $c_1, \dots, c_\ell$ which in turn determine pairs (β_i, α_i) for $0 \leq i \leq \ell+1$. As Wemyss [14, Lemma 3.5] remarks, the next combinatorial result follows from Riemenschneider's staircase [12].

Lemma 4.9. *Set* $m_i := \max\{1, c_i - 1\}$ *for* $1 \leq i \leq \ell$ *and* $m_{\ell+1} = 1$*. The inequalities*

$$\alpha_{i+1} - m_i\alpha_i > \alpha_i - \alpha_{i-1} \quad \text{and} \quad m_i\beta_i - \beta_{i+1} < \beta_{i-1} - \beta_i \tag{4.2}$$

hold for every $1 \leq i \leq \ell$*, and the minimal* $\Bbbk$*-algebra generators of* $\Bbbk[x, y]^G$ *are*

$$\{y^r\} \cup \bigcup_{0\leq i\leq \ell} \{x^{\alpha_{i+1}-t_i\alpha_i} y^{t_i\beta_i-\beta_{i+1}} : 1 \leq t_i \leq m_i\} \cup \{x^r\}. \tag{4.3}$$

In particular, listing the monomials from (4.3) *with strictly decreasing exponent of x is equivalent to listing the monomials with strictly increasing exponent of y.*

Consider elements of $Q_0 = \{0, 1, , \dots, \ell\}$ modulo $\ell + 1$, and write $\bar{i} := i \mod \ell + 1$. Recall also that $(\beta_0, \alpha_0) = (r, 0)$ and $(\beta_{\ell+1}, \alpha_{\ell+1}) = (0, r)$.

Proposition 4.10. *For each* $i \in Q_0 \setminus \{0\}$ *there are precisely two arrows with head at* i*:*

(i) *one arrow, denoted* a_{2i-1}*, from* $i-1$ *to* i *has monomial label* $\operatorname{mon}(a_{2i-1}) = x^{\alpha_i - \alpha_{i-1}}$*;*

(ii) *the other, denoted* a_{2i+2}*, from* $\overline{i+1}$ *to* i *has monomial label* $\operatorname{mon}(a_{2i+2}) = y^{\beta_i - \beta_{i+1}}$*.*

In particular, any arrow $a \in Q_1$ *with* $xy | \operatorname{mon}(a)$ *has its head* $\mathrm{h}(a)$ *at vertex* 0*.*

Proof. For $i \in Q_0 \setminus \{0\}$, there are at least two arrows with head at i since the $H^0(\mathscr{O}_X)$-algebra generators $x^{\alpha_i}, y^{\beta_i}$ of $H^0(L_i)$ define paths in Q from 0 to i with monomial a pure power of x and a pure power of y respectively. The path labelled x^{α_i} passes through vertex $i-1$, so the final arrow a_{2i-1} in this path has $\operatorname{mon}(a_{2i-1}) = x^{\alpha_i - \alpha_{i-1}}$. Similarly, the path labelled y^{β_i} passes through $\overline{i+1}$, so the final arrow a_{2i+2} in this path has label $y^{\beta_i - \beta_{i+1}}$. It remains to show that for $a \in Q_1$ with $\mathrm{h}(a) \in Q_0 \setminus \{0\}$, the monomial $\operatorname{mon}(a)$ is not divisible by xy.

Suppose otherwise, so $\operatorname{mon}(a) = x^b y^c$ for $b, c \geq 1$. Set $i = \mathrm{h}(a)$ and $j := \mathrm{t}(a)$. The path $\iota(a)$ in the McKay quiver Q' from $\iota(\mathrm{t}(a))$ to $\iota(\mathrm{h}(a))$ can be chosen to comprise b arrows labelled x followed by c arrows labelled y. If $b \geq \alpha_{j+1} - \alpha_j$ then this path begins with sufficiently many arrows labelled x to factor via the unique path p' in Q' satisfying $\iota(a_{2j+1}) = p'$, but this would imply that the path a in Q factors via the arrow a_{2j+1} which is absurd. Also, if $c \geq \beta_i - \beta_{i+1}$ then $\iota(a)$ ends by traversing the unique path q' in Q' that satisfies $\iota(a_{2i+2}) = q'$, but this would force the path a to factor via arrow a_{2i+2} which is also absurd. Similarly, the path $\iota(a)$ may also be chosen to comprise c arrows labelled y followed by b arrows labelled x. If $c \geq \beta_{j-1} - \beta_j$ then this path in Q' begins with sufficiently many arrows labelled y to ensure that the path a in Q factors via the arrow a_{2j} which is absurd. Also, if $b \geq \alpha_i - \alpha_{i-1}$ then the path $\iota(a)$ in Q' ends with sufficiently many arrows labelled x to ensure that the path a in Q factors via the arrow a_{2i-1} which is also absurd. Taken together, then, we obtain inequalities

$$1 \leq b < \min(\alpha_{j+1} - \alpha_j, \alpha_i - \alpha_{i-1}) \quad \text{and} \quad 1 \leq c < \min(\beta_i - \beta_{i+1}, \beta_{j-1} - \beta_j). \tag{4.4}$$

We now demonstrate that no monomial $x^b y^c$ satisfying the inequalities (4.4) lies in the character space $\iota(i) \otimes \iota(j)^{-1}$. There are three cases:

Case 1: $i > j$. In this case we have $\alpha_{j+1} \leq \alpha_i$. Multiply the generator $x^{\alpha_j} \in H^0(L_j)$ by $x^b y^c$ to obtain $x^{b+\alpha_j} y^c \in H^0(L_i)$. The inequalities (4.4) give $b + \alpha_j < \alpha_{j+1} \leq \alpha_i$ and $c < \beta_i - \beta_{i+1} < \beta_i$. However, this implies that the section $x^{b+\alpha_j} y^c$ is divisible by neither $H^0(\mathscr{O}_X)$-algebra generator $x^{\alpha_i}, y^{\beta_i}$ of $H^0(L_i)$ which is a contradiction.

Case 2: $i = j$. In this case, $x^b y^c$ is G-invariant and is therefore divisible by a G-invariant monomial $x^{\alpha_{k+1} - t_k \alpha_k} y^{t_k \beta_k - \beta_{k+1}}$ from the list (4.3). Then (4.4) gives $t_k \beta_k - \beta_{k+1} \leq c < \beta_i - \beta_{i+1}$, which forces $k > i$. On the other hand, inequalities (4.4) also give $\alpha_{k+1} - t_k \alpha_k \leq b < \alpha_i - \alpha_{i-1}$, from which we obtain $k < i$. This gives the required contradiction.

Case 3: $i < j$. In this case we have $\beta_{j-1} \leq \beta_i$. Multiply the generator $y^{\beta_j} \in H^0(L_j)$ by $x^b y^c$ to obtain $x^b y^{c+\beta_j} \in H^0(L_i)$. The inequalities (4.4) give $c + \beta_j < \beta_{j-1} \leq \beta_i$ and $b < \alpha_i - \alpha_{i-1} < \alpha_i$, but then $x^b y^{c+\beta_j}$ is divisible by neither $H^0(\mathscr{O}_X)$-algebra generator $x^{\alpha_i}, y^{\beta_i}$ of $H^0(L_i)$ which is a contradiction.

In each case we obtain a contradiction, so no such arrow exists. This completes the proof. $\square$

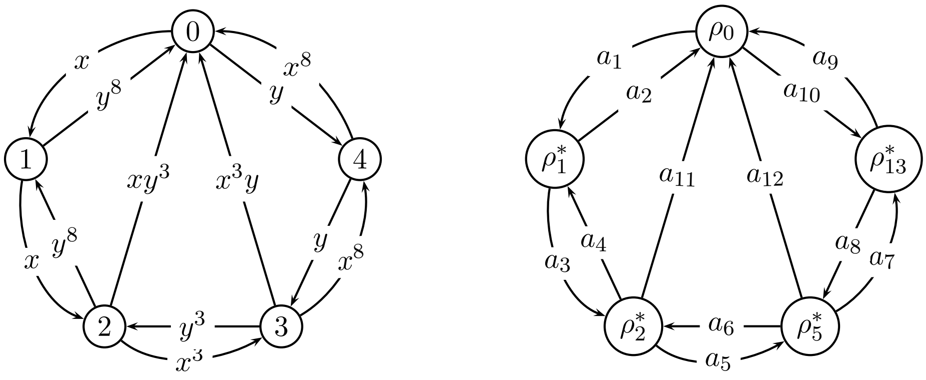
Remark 4.11. Since x^r arises as the label on a cycle in Q' based at 0, there exists one arrow a in Q from ℓ to 0 with $\operatorname{mon}(a)$ equal to a pure power of x. Similarly, y^r labels a cycle in Q' based at 0, so Q contains one arrow a from 1 to 0 with $\operatorname{mon}(a)$ equal to a pure power of y.

We may therefore list the arrow set as $Q_1 = \{a_1, \ldots, a_{|Q_1|}\}$, comprising:

- the x-*arrows* $a_1, a_3, \ldots, a_{2\ell+1}$, where a_{2i-1} has tail at $i-1$, head at i and label $x^{\alpha_i-\alpha_{i-1}}$;
- the y-*arrows* $a_2, a_4, \ldots, a_{2\ell+2}$, where a_{2i+2} has tail at $\overline{i+1}$, head at i and label $y^{\beta_i-\beta_{\overline{i+1}}}$;
- the xy-*arrows* $a_{2\ell+3}, a_{2\ell+4} \ldots, a_{|Q_1|}$ each with head at 0 and $\mathrm{mon}(a_{2\ell+k})$ divisible by xy.

The x-arrows and y-arrows pair up naturally as $\{a_{2i+1}, a_{2i+2}\}$ for $0 \leq i \leq \ell$. The order in which the xy-arrows are listed is not important for now, so choose any order.

Example 4.12. For the action of type $\frac{1}{21}(1, 13)$, the continued fraction 21/13 defines the pairs $(\beta_5, \alpha_5) = (0, 21)$, $(\beta_4, \alpha_4) = (1, 13)$, $(\beta_3, \alpha_3) = (2, 5)$, $(\beta_2, \alpha_2) = (5, 2)$, $(\beta_1, \alpha_1) = (13, 1)$ and $(\beta_0, \alpha_0) = (21, 0)$. The special representations are $\rho_1^*, \rho_2^*, \rho_5^*, \rho_{13}^*$ and the monomials labelling arrows in the Special McKay quiver are shown in Figure 3. The $\Bbbk$-algebra generators of $\Bbbk[x, y]^G$ from (4.3) are $y^{21}, xy^8, x^3y^3, x^8y, x^{21}$.



(a) Labelling monomials (b) Listing the arrows

FIGURE 3. The Special McKay quiver of type $\frac{1}{21}(1, 13)$

4.3. **The moduli space is irreducible.** We now prove that the fine moduli space $\mathcal{M}_\vartheta(Q, R) = \mathbb{V}(I_R) /\!\!/_\vartheta T_Q$ of ϑ-stable representations of the bound Special McKay quiver is isomorphic to the image of the morphism $\varphi_{|\underline{\mathscr{L}}|} \colon X \to |\underline{\mathscr{L}}|$ from Theorem 3.4. We bypass the explicit calculation of the ideal I_R by introducing an auxilliary ideal $J \subset \Bbbk[y_a : a \in Q_1]$ which suffices for our purpose.

As a first step we exhibit a collection of elements from the ideal R. A cycle p in Q is said to be *primitive* if $\mathrm{mon}(p)$ is one of the minimal $\Bbbk$-algebra generators of $\Bbbk[x, y]^G$ from (4.3).

Lemma 4.13. *For $i \in Q_0 \setminus \{0\}$, consider an arrow $a \in Q_1$ with $\mathrm{t}(a) = i$. Then either:*

(1) *$a = a_{2i+1}$, in which case there exists a primitive cycle p_a in Q based at vertex i such that $a_{2i+1}a_{2i+2} - p_a \in R$ with $a_{2i+1}, a_{2i+2} \notin \mathrm{supp}(p_a)$;*

(2) *$a = a_{2i}$, in which case there exists a primitive cycle p_a in Q based at vertex i such that $a_{2i}a_{2i-1} - p_a \in R$ with $a_{2i-1}, a_{2i} \notin \mathrm{supp}(p_a)$;*

(3) *a is an xy-arrow, in which case there exists both:*
   (i) *a primitive cycle p_a in Q based at vertex i satisfying $aa_1a_3 \cdots a_{2i-1} - p_a \in R$ and $a, a_1, a_3, \ldots, a_{2i-1} \notin \mathrm{supp}(p_a)$; and*
   (ii) *a primitive cycle q_a in Q based at vertex i satisfying $aa_{2\ell+2}a_{2\ell} \cdots a_{2i+2} - q_a \in R$ and $a, a_{2\ell+2}, a_{2\ell}, \ldots, a_{2i+2} \notin \mathrm{supp}(q_a)$.*

Proof. For case (1), set $a = a_{2i+1}$. Since a is an x-arrow and a_{2i+2} is a y-arrow, Proposition 4.7 implies that the cycle aa_{2i+2} in Q based at i arises from a cycle $p' := \iota(aa_{2i+2})$ in Q' based at $\iota(i) \in Q'_0$ where $\mathrm{mon}(p') = x^m y^n$ with $m, n \geq 1$. Since a precedes a_{2i+2}, the first arrow $a'_1 \in Q'_1$ traversed by p' satisfies $\mathrm{mon}(a'_1) = x$. Define p'_a in Q' to be the unique cycle based at $\iota(i) \in Q'_0$ with $\mathrm{mon}(p'_a) = x^m y^n$ that first traverses n arrows labelled y and then traverses m arrows labelled x. The unique cycle p_a in Q satisfying $p'_a = \iota(p_a)$ first traverses an arrow a_f with $\mathrm{t}(a_f) = i$ and $y | \mathrm{mon}(a_f)$ and ends by traversing an arrow a_l with $\mathrm{h}(a_l) = i$ and $x | \mathrm{mon}(a_l)$. Proposition 4.10 implies that $a_l = a_{2i-1}$. If a_f is a y-arrow then $a_f = a_{2i}$ and $p_a = a_{2i}a_{2i-1}$ is the only such primitive cycle; otherwise, a_f is an xy-arrow, in which case $p_a = a_f a_1 a_3 \cdots a_{2i-1}$ is the only such primitive cycle. In either case $a_{2i+1}, a_{2i+2} \notin \mathrm{supp}(p_a)$, so case (1) is complete. Case (2) is the same as case (1) with the roles of x and y switched. For case (3), let a be an xy-arrow, so $\mathrm{t}(a) = i$, $\mathrm{h}(a) = 0$ and $\mathrm{mon}(a) = x^m y^n$ for $m, n \geq 1$. The cycle $p' := \iota(aa_1a_3 \cdots a_{2i-1})$ in Q' based at $\iota(i) \in Q'_0$ ends with at least α_i arrows labelled x and satisfies $\mathrm{mon}(p') = x^{m+\alpha_i} y^n$. Define p'_a in Q' to be the unique cycle based at $\iota(i) \in Q'_0$ with $\mathrm{mon}(p'_a) = x^{m+\alpha_i} y^n$ that first traverses $m + \alpha_i$ arrows labelled x and then traverses n arrows labelled y. The unique cycle p_a in Q satisfying $p'_a = \iota(p_a)$ first traverses an arrow a_f with $\mathrm{t}(a_f) = i$ and $x | \mathrm{mon}(a_f)$, and ends by traversing an arrow a_l with $\mathrm{h}(a_l) = i$ and $y | \mathrm{mon}(a_l)$. Proposition 4.10 gives $a_l = a_{2i+2}$. If a_f is an x-arrow then $a_f = a_{2i+1}$ and $p_a = a_{2i+1}a_{2i+2}$ is the only such primitive cycle; otherwise, a_f is an xy-arrow, in which case $p_a = a_f a_{2\ell+2} a_{2\ell} \cdots a_{2i+2}$ is the only such primitive cycle. Note that $a_f \neq a$ because $x^{\alpha_i} = \mathrm{mon}(a_1 a_3 \cdots a_{2i-1}) \neq \mathrm{mon}(a_{2\ell+2} a_{2\ell} \cdots a_{2i+2}) = y^{\beta_i}$. It follows that in either case $a, a_1, a_3, \ldots, a_{2i-1} \notin \mathrm{supp}(p_a)$, so case (3.i) is complete. Case (3.ii) is similar. $\square$

Let Λ denote the set of all path differences constructed in Lemma 4.13 as the vertex i ranges over $Q_0 \setminus \{0\}$ and the arrow a ranges over $\{a \in Q_1 : \mathrm{t}(a) = i\}$. Define the ideal

$$J := \big(y_p - y_q \in \Bbbk[y_a : a \in Q_1] : p - q \in \Lambda\big).$$

Remark 3.7 and Lemma 4.13 imply that $J \subseteq I_R \subseteq I_Q$, and hence $\mathbb{V}(I_Q) \subseteq \mathbb{V}(I_R) \subseteq \mathbb{V}(J)$.

Remark 4.14. We make no attempt to prove that $J = I_R$ since this requires an understanding of R itself. The explicit calculation of R by Wemyss [14, pages 3-14] does in fact imply that $J = I_R$, but we emphasise that we do not make use of this observation. Rather, our approach is to show that while $\mathbb{V}(J) \neq \mathbb{V}(I_Q)$, we nevertheless get equality after removing the locus cut out by the irrelevant ideal, that is, $\mathbb{V}(J) \setminus \mathbb{V}(B_Q) = \mathbb{V}(I_R) \setminus \mathbb{V}(B_Q) = \mathbb{V}(I_Q) \setminus \mathbb{V}(B_Q)$. We illustrate this with a pair of examples.

Example 4.15. For the action of type $\frac{1}{7}(1,2)$, the quiver Q from Figure 1(b) is obtained from the McKay quiver Q' from Figure 2(b) by keeping only paths with head and tail at the vertices indexed by $\mathscr{O}_X = \mathscr{W}_{\rho_0^*}$, $L_1 = \mathscr{W}_{\rho_1^*}$ and $L_2 = \mathscr{W}_{\rho_2^*}$. To obtain the monomials $\mathrm{mon}(a)$ for $a \in Q_1$, consider the labels in Figure 1(b) and replace x_0 by x, x_3 by y and set $x_1 = x_2 = 1$. Lemma 4.13 constructs the ideal J as follows: arrows a_2 and a_3 with tail at 1 each determine $y_1y_2 - y_3y_4 \in \Lambda$; the x-arrow a_5 determines $y_5y_6 - y_1y_3y_7$; the y-arrow a_4 determines $y_3y_4 - y_6y_8$; the xy-arrow a_7 determines both $y_1y_3y_8 - y_6y_7$ and $y_6y_8 - y_3y_4$; and the xy-arrow a_8 determines

both $y_6y_7 - y_1y_3y_8$ and $y_1y_3y_7 - y_5y_6$. Taken together we obtain

$$\begin{aligned} J &= \big(y_1y_2 - y_3y_4,\ y_5y_6 - y_1y_3y_7,\ y_3y_4 - y_6y_8,\ y_1y_3y_8 - y_6y_7\big), \\ &= I_Q \cap (y_1, y_4, y_6) \cap (y_1, y_3, y_6) \cap (y_2, y_3, y_6). \end{aligned}$$

Saturating by the irrelevant ideal $B_Q = (y_1y_3, y_1y_6, y_4y_6)$ gives $J : B_Q = I_Q : B_Q$, and hence $\mathbb{V}(J) \setminus \mathbb{V}(B_Q) = \mathbb{V}(I_Q) \setminus \mathbb{V}(B_Q)$.

Example 4.16. The Special McKay quiver for the action of type $\frac{1}{21}(1, 13)$ is shown in Figure 3. Applying Lemma 4.13 repeatedly gives

$$J = \left(\begin{array}{c} y_1y_2 - y_3y_4,\ y_7y_8 - y_9y_{10},\ y_8y_{10}y_{12} - y_5y_6, \\ y_1y_3y_{11} - y_5y_6,\ y_1y_3y_5y_{12} - y_9y_{10},\ y_6y_8y_{10}y_{11} - y_3y_4 \end{array} \right).$$

This ideal has 25 primary components, one of which is the toric ideal I_Q and 24 of which cut out linear varieties. Saturating by the irrelevent ideal

$$B_Q = (y_1y_3y_5y_7,\ y_1y_3y_5y_{10},\ y_1y_3y_8y_{10},\ y_1y_6y_8y_{10},\ y_4y_6y_8y_{10})$$

gives $J : B_Q = I_Q : B_Q$, from which we obtain $\mathbb{V}(J) \setminus \mathbb{V}(B_Q) = \mathbb{V}(I_Q) \setminus \mathbb{V}(B_Q)$.

Theorem 4.17. *The fine moduli space $\mathcal{M}_\vartheta(Q, R)$ of ϑ-stable representations of the bound Special McKay quiver is isomorphic to the toric variety $\mathbb{V}(I_Q)/\!\!/_\vartheta T_Q$.*

Proof. Proposition 3.3 implies that $\mathbb{A}^{Q_1}_\Bbbk \setminus \mathbb{V}(B_Q)$ is covered by charts $\operatorname{Spec}\big(\Bbbk[y_a : a \in Q_1]_{y_\mathcal{T}}\big)$, where $\mathcal{T} \subseteq Q_1$ is a spanning tree with root at $0 \in Q_0$ and $y_\mathcal{T} := \prod_{a \in \operatorname{supp}(\mathcal{T})} y_a$. The inclusion $J \subseteq I_Q$ implies that $\mathbb{V}(I_Q) \setminus \mathbb{V}(B_Q)$ is a closed subscheme of $\mathbb{V}(J) \setminus \mathbb{V}(B_Q)$, and restricting to the open cover gives $\operatorname{Spec}\big((\Bbbk[y_a : a \in Q_1]/I_Q)_{y_\mathcal{T}}\big) \subseteq \operatorname{Spec}\big((\Bbbk[y_a : a \in Q_1]/J)_{y_\mathcal{T}}\big)$ for each $\mathcal{T}$. The variety $\mathbb{V}(I_Q)/\!\!/_\vartheta T_Q$ is the geometric quotient of $\mathbb{V}(I_Q) \setminus \mathbb{V}(B_Q)$ by the action of T_Q, and since each toric chart of $\mathbb{V}(I_Q)/\!\!/_\vartheta T_Q \cong X$ is isomorphic to $\mathbb{A}^2_\Bbbk$, it follows that $\operatorname{Spec}\big((\Bbbk[y_a : a \in Q_1]/I_Q)_{y_\mathcal{T}}\big)$ is isomorphic to $T_Q \times \mathbb{A}^2_\Bbbk$ for every spanning tree $\mathcal{T}$. We claim that $\operatorname{Spec}\big((\Bbbk[y_a : a \in Q_1]/J)_{y_\mathcal{T}}\big)$ is also isomorphic to $T_Q \times \mathbb{A}^2_\Bbbk$ for every $\mathcal{T}$, giving $\mathbb{V}(I_Q) \setminus \mathbb{V}(B_Q) = \mathbb{V}(J) \setminus \mathbb{V}(B_Q)$. The result follows from the claim, because the inclusions $J \subseteq I_R \subseteq I_Q$ force $\mathbb{V}(I_R) \setminus \mathbb{V}(B_Q) = \mathbb{V}(I_Q) \setminus \mathbb{V}(B_Q)$, and the required identification $\mathcal{M}_\vartheta(Q, R) := \mathbb{V}(I_R)/\!\!/_\vartheta T_Q = \mathbb{V}(I_Q)/\!\!/_\vartheta T_Q$ follows from Proposition 3.3.

To prove the claim we first list all spanning trees $\mathcal{T}$ in Q with root at $0 \in Q_0$. Proposition 4.10 implies that there are $\ell + 1$ such trees, namely, $\mathcal{T}_0, \dots, \mathcal{T}_\ell$ where $\operatorname{supp}(\mathcal{T}_0) = \{a_4, a_6, \dots, a_{2\ell+2}\}$, $\operatorname{supp}(\mathcal{T}_j) = \{a_1, a_3, \dots, a_{2j-1}\} \cup \{a_{2j+4}, a_{2j+6}, \dots, a_{2\ell+2},\}$ for every $0 < j < \ell$, and $\operatorname{supp}(\mathcal{T}_\ell) = \{a_1, a_3, \dots, a_{2\ell-1}\}$. The result follows once we prove that

$$\Big(\frac{\Bbbk[y_a : a \in Q_1]}{J}\Big)_{y_{\mathcal{T}_j}} \cong \Bbbk[y_a^{\pm 1} : a \in \operatorname{supp}(\mathcal{T}_j)] \otimes_\Bbbk \Bbbk[y_{2j+1}, y_{2j+2}] \quad \text{for } 0 \le j \le \ell. \tag{4.5}$$

Fix $0 \le j \le \ell$. We use the ideal J to eliminate certain variables y_a in two stages (if $j = 0$ ignore stage one, and if $j = \ell$ ignore stage two):

(1) *eliminate each y_a with $1 \le \mathrm{t}(a) \le j$ and $y|\operatorname{mon}(a)$.* Fix $i \in Q_0 \setminus \{0\}$ with $i \le j$ and consider the y-arrow a_{2i}. Lemma 4.13(2) gives $y_{2i} = y_{p_a} y_{2i-1}^{-1}$ in $\big(\Bbbk[y_a : a \in Q_1]/J\big)_{y_\mathcal{T}}$ where y_{p_a} is either $y_{2i+1}y_{2i+2}$ or $y_{a_f}y_{2\ell+2} \cdots y_{2i+2}$ for some xy-arrow a_f with $\mathrm{t}(a_f) = i$. In either case, y_{2i} is expressed in terms of variables with a higher index, and we eliminate it. As for the xy-arrows with tail at i, choose one such with $\operatorname{mon}(a)$ divisible by the lowest

power of x. Lemma 4.13(3.i) gives $y_a = y_{p_a} \cdot (y_1 y_3 \cdots y_{2i-1})^{-1}$ in $\big(\Bbbk[y_a : a \in Q_1]/J\big)_{y_T}$, where y_{p_a} is either $y_{2i+1}y_{2i+2}$ or $y_{a_f}y_{2\ell+2}\dots y_{2i+2}$ for some xy-arrow a_f with tail at i and $\mathrm{mon}(a_f)$ is divisible by a strictly higher power of x than is $\mathrm{mon}(a)$. In either case, y_a is expressed in variables indexed by arrows from $\{a_1, \dots, a_{2i+1}\} \cup \{a_{2i+2}, \dots, a_{2\ell+2}\}$ and xy-arrows b with tail at i and $\mathrm{mon}(b)$ divisible by a strictly higher power of x than was $\mathrm{mon}(a)$. Repeat, until each y_a indexed by an xy-arrow a with tail at i is expressed in variables $\{y_1, \dots, y_{2i+1}\} \cup \{y_{2i+2}, \dots, y_{2\ell+2}\}$. Carry out this procedure, in order, at the vertices on the list $(1, 2, \dots, j)$. The result is that each y_a with $1 \leq \mathrm{t}(a) \leq j$ and $y|\,\mathrm{mon}(a)$ is written in terms of $\{y_1, y_3, \dots, y_{2j+1}\} \cup \{y_{2j+2}, \dots, y_{2\ell+2}\}$.

(2) *eliminate each y_a with $j+1 \leq \mathrm{t}(a) \leq \ell$ and $x|\,\mathrm{mon}(a)$.* Fix $i \geq j+1$ and consider the x-arrow a_{2i+1}. Lemma 4.13(1) gives $y_{2i+1} = y_{p_a} y_{2i+2}^{-1}$ in $\big(\Bbbk[y_a : a \in Q_1]/J\big)_{y_T}$ where y_{p_a} is either $y_{2i-1}y_{2i}$ or $y_{a_f}y_1 \dots y_{2i-1}$ for some xy-arrow a_f with tail at i. In either case, y_{2i+1} is expressed in terms of x-arrows $\{a_1, a_3 \cdots a_{2i-1}\}$, the y-arrows $\{a_{2i}, a_{2i+2}\}$ and an xy-arrow with tail at i, so we eliminate it. As for the xy-arrows with tail at i, choose one such with $\mathrm{mon}(a)$ divisible by the lowest power of y. Lemma 4.13(3.ii) gives $y_a = y_{q_a} \cdot (y_{2\ell+2}y_{2\ell} \cdots y_{2i+2})^{-1}$ in $\big(\Bbbk[y_a : a \in Q_1]/J\big)_{y_T}$, where y_{q_a} is either $y_{2i-1}y_{2i}$ or $y_{a_f}y_1 \dots y_{2i-1}$ for some xy-arrow a_f with tail at i and $\mathrm{mon}(a_f)$ is divisible by a strictly higher power of y than is $\mathrm{mon}(a)$. In either case, y_a is expressed in variables indexed by arrows from $\{a_1, \dots, a_{2i-1}\} \cup \{a_{2i}, \dots, a_{2\ell+2}\}$ and xy-arrows b with tail at i and $\mathrm{mon}(b)$ divisible by a higher power of y than was $\mathrm{mon}(a)$. Repeat, until each y_a indexed by an xy-arrow a with tail at i is expressed in variables $\{y_1, \dots, y_{2i-1}\} \cup \{y_{2i}, \dots, y_{2\ell+2}\}$. Carry out this procedure, in order, at the vertices on the list $(\ell, \ell-1, \dots, j+1)$, so each y_a with $j+1 \leq \mathrm{t}(a) \leq \ell$ and $x|\,\mathrm{mon}(a)$ is expressed in $\{y_1, y_3, \dots, y_{2j+1}\} \cup \{y_{2j+2}, \dots, y_{2\ell+2}\}$.

Applying Stages 1 and 2 gives each variable y_a from $\big(\Bbbk[y_a : a \in Q_1]/J\big)_{y_T}$ in terms of the variables $\{y_a : a \in \mathrm{supp}(\mathcal{T}_j)\} \cup \{y_{2j+1}, y_{2j+2}\}$. It follows that the $\Bbbk$-algebra homomorphism

$$\Bbbk[y_a^{\pm 1} : a \in \mathrm{supp}(\mathcal{T}_j)] \otimes_\Bbbk \Bbbk[y_{2j+1}, y_{2j+2}] \longrightarrow \big(\Bbbk[y_a : a \in Q_1]/J\big)_{y_T}$$

sending y_a to $[y_a \mod J]$ is an epimorphism. If the kernel is nonzero then the affine chart $\mathrm{Spec}\,\big((\Bbbk[y_a : a \in Q_1]/J)_{y_\mathcal{T}}\big)$ is a proper subscheme of $T_Q \times \mathbb{A}^2_\Bbbk \cong \mathrm{Spec}\,\big(\Bbbk[y_a^{\pm 1} : a \in \mathrm{supp}(\mathcal{T}_j)] \otimes_\Bbbk \Bbbk[y_{2j+1}, y_{2j+2}]\big)$, contradicting the inclusion $\mathbb{V}(I_Q) \setminus \mathbb{V}(B_Q) \subseteq \mathbb{V}(J) \setminus \mathbb{V}(B_Q)$. □

5. The Special McKay correspondence following Van den Bergh

To conclude we consider the derived category. Motivated by Bridgeland's work on perverse coherent sheaves, Van den Bergh [13] considered the subcategory $\mathcal{B} :=^{-1} \mathrm{Per}\big(X/(\mathbb{A}^2_\Bbbk/G)\big)$ of the bounded derived category of coherent sheaves on X consisting of those objects E whose cohomology sheaves $\mathcal{H}^i(E)$ are nonzero only in degrees -1 and 0, such that $f_*(\mathcal{H}^{-1}(E)) = 0$, and such that the map $f^*f_*\mathcal{H}^0(E) \to \mathcal{H}^0(E)$ is surjective. A *projective generator* for $\mathcal{B}$ is defined to be a projective object P in $\mathcal{B}$ for which $\mathrm{Hom}_\mathcal{B}(P, E) = 0$ implies $E = 0$.

Lemma 5.1. *The vector bundle $\bigoplus_{i \in Q_0} L_i$ on X is a projective generator for $\mathcal{B}$.*

Proof. The minimal resolution satisfies $\mathbf{R}f_*(\mathscr{O}_X) = \mathscr{O}_{\mathbb{A}^2_\Bbbk/G}$. Since $\Bbbk$ is algebraically closed of characteristic zero, and since each fibre of f has dimension at most one, the results from Van

den Bergh [13, Section 3] hold. In particular, since the ample line bundle $L = \bigotimes_{i=1}^{\ell} L_i$ satisfies $L = \det(\bigoplus_{i=1}^{\ell} L_i)$, the Artin–Verdier construction gives a short exact sequence

$$0 \longrightarrow \mathscr{O}_X^{\oplus(\ell-1)} \longrightarrow \bigoplus_{i=1}^{\ell} L_i \longrightarrow L \longrightarrow 0$$

by [13, Lemma 3.5.1]. Since L is globally generated, the proof of [13, Proposition 3.2.5] shows that the bundle $\mathscr{O}_X \oplus \bigoplus_{i=1}^{\ell} L_i$ is a projective generator for $\mathcal{B}$. □

For $A = \operatorname{End}\big(\bigoplus_{i\in Q_0} L_i\big)$, write $\operatorname{mod}(A)$ for the category of finitely generated right A-modules.

Proof of Theorem 1.2. Since $\bigoplus_{i\in Q_0} L_i$ on X is a projective generator for $\mathcal{B}$, the construction [13, Corollary 3.2.8] gives an equivalence between $\mathcal{B}$ and $\operatorname{mod}(A)$ that extends to an equivalence

$$\mathbf{R}\operatorname{Hom}\big(\bigoplus_{i\in Q_0} L_i, -\big)\colon D^b\big(\operatorname{Coh}(X)\big) \longrightarrow D^b\big(\operatorname{mod}(A)\big)$$

of derived categories. This completes the proof of Theorem 1.2. □

References

[1] T. Bridgeland, A. King, and M. Reid. The McKay correspondence as an equivalence of derived categories. *J. Amer. Math. Soc.*, 14(3):535–554 (electronic), 2001.

[2] D. Cox. The homogeneous coordinate ring of a toric variety. *J. Algebraic Geom.*, 4(1):17–50, 1995.

[3] A. Craw, D. Maclagan, and R.R. Thomas. Moduli of McKay quiver representations I: the coherent component. *Proc. London Math. Soc.*, **95**(1):179–198, 2007.

[4] A. Craw, D. Maclagan, and R.R. Thomas. Moduli of McKay quiver representations. II. Gröbner basis techniques. *J. Algebra*, 316(2):514–535, 2007.

[5] A. Craw and G. G. Smith. Projective toric varieties as fine moduli spaces of quiver representations. *Amer. J. Math.*, 130(6):1509–1534, 2008.

[6] D. Grayson and M. Stillman. Macaulay 2, a software system for research in algebraic geometry. Available from `http://www.math.uiuc.edu/Macaulay2/`.

[7] A. Ishii. On the McKay correspondence for a finite small subgroup of $\mathrm{GL}(2, \mathbb{C})$. *J. Reine Angew. Math.*, 549:221–233, 2002.

[8] Y. Ito and I. Nakamura. Hilbert schemes and simple singularities. In *New trends in algebraic geometry (Warwick, 1996)*, volume 264 of *London Math. Soc. Lecture Note Ser.*, pages 151–233. Cambridge Univ. Press, Cambridge, 1999.

[9] M. Kapranov and E. Vasserot. Kleinian singularities, derived categories and Hall algebras. *Math. Ann.*, 316:565–576, 2000.

[10] R Kidoh. Hilbert schemes and cyclic quotient surface singularities. *Hokkaido Math. J.*, **30**(1):91–103, (2001).

[11] A. King. Moduli of representations of finite-dimensional algebras. *Quart. J. Math. Oxford Ser. (2)*, **45**(180):515–530, 1994.

[12] O. Riemenschneider. Deformationen von Quotientensingularitäten (nach zyklischen Gruppen). *Math. Ann.*, 209:211–248, 1974.

[13] M. Van den Bergh. Three-dimensional flops and noncommutative rings. *Duke Math. J.*, 122(3):423–455, 2004.

[14] M. Wemyss. Reconstruction algebras of type A, 2007. `arXiv:0704.3693`, to appear *Trans. Amer. Math. Soc.*

[15] J. Wunram. Reflexive modules on cyclic quotient surface singularities. In *Singularities, representation of algebras, and vector bundles (Lambrecht, 1985)*, volume 1273 of *Lecture Notes in Mathematics*, pages 221–231. Springer, Berlin, 1987.

[16] J. Wunram. Reflexive modules on quotient surface singularities. *Math. Ann.*, **279**(4):583–598, (1988).

Department of Mathematics, Glasgow University, Glasgow G12 8QW, Scotland.
E-mail address: `craw@maths.gla.ac.uk`